NORTH-HOLLAND

p-Competition Graphs

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Dedicated to Professor John Maybee on the occasion of his 65th birthday.

Submitted by J. Richard Lundgren

ABSTRACT

If $D = (V, A)$ is a digraph, its p-competition graph has vertex set V and an edge between x and y if and only if there are distinct vertices $a_1, \ldots, a_p$

LINEAR ALGEBRA AND ITS APPLICATIONS 217:167–178 (1995)

0024-3795/95/$9.50
SSDI 0024-3795(94)00060-Q

and arcs (x, a_i) and (y, a_i) in D for each $i \leq p$. This definition generalizes the widely studied $p = 1$ case of ordinary competition graphs. We obtain results about p-competition graphs analogous to the well-known results about ordinary competition graphs and apply these results to specific cases.

1. INTRODUCTION

The notion of competition graph was introduced by Joel Cohen [7] in connection with a problem in ecology. Since Cohen's initial article, an extensive literature has appeared on competition graphs and their various applications, which in addition to ecology include applications to channel assignments, coding, and modeling of complex economic and energy systems; see [31]. For surveys of the literature of competiton graphs, see [19, 23, 39]. Many variations of ordinary competition graph have been introduced, including the common enemy graph (resource graph) in [26, 38], the competition–common enemy graph in [18, 22, 36, 37], the niche graph in [3, 5, 10], and the competition multigraph in [1].

We introduced another such generalization, the p-competition graph, in [20]. This notion and related parameters have spawned considerable interest, including [11, 14, 17, 21]. The present paper contains the foundations of that work, as first presented in [20].

Suppose $D = (V, A)$ is a digraph. (For all undefined graph theory terminology, see Bondy and Murty [2] or Roberts [33].) If p is a positive integer, the *p-competition graph* $C_p(D)$ corresponding to D is defined to have vertex set V with an edge between x and y in V if and only if, for some distinct $a_1, \ldots, a_p$ in V, the pairs $(x, a_1), (y, a_1), (x, a_2), (y, a_2), \ldots, (x, a_p), (y, a_p)$ are arcs. If D is thought of as a food web whose vertices are species in some ecosystem, with an arc (x, y) if and only if x preys on y, then $\{x, y\}$ is an edge of $C_p(D)$ if and only if x and y have at least p common prey. Note that $C_1(D)$ is the ordinary *competition graph* of [7].

It is common in the literature of competition graphs to make special assumptions about D, in particular that it is acyclic. However, the definition makes sense in general. The definition can be thought of as a special case of a more general notion of tolerance intersection graph which has been developed in [15, 16].

In the present paper, we study the properties of p-competition graphs, obtaining, where possible, analogues of results about ordinary competition graphs. Section 2 introduces p-edge clique coverings as the basic tool for the study of p-competition graphs of arbitrary digraphs (loops allowed), producing results analogous to those of Dutton and Brigham [9]. We also present several basic results for 2-competition graphs. Section 3 studies p-competition graphs of loopless digraphs, producing results analogous to those of Roberts and Steif [35], and also the acyclic case, producing results analogous to those of Dutton and Brigham [9] and Lundgren and Maybee [24]. Section 4 contains closing remarks and in particular applies our methods to an analogue of the notion of upper bound graph, obtaining a result analogous to those of McMorris and Zaslavsky [30].

2. p-COMPETITION GRAPHS OF ARBITRARY DIGRAPHS

In this section, we study p-competition graphs of arbitrary digraphs. Following the work on ordinary competition graphs in [9], we allow loops in this section.

The notion of edge clique covering plays an important role in the study of ordinary competition graphs. An *edge clique covering*, or *ECC*, of G is a collection of cliques such that every edge of G is in at least one of these cliques. In the case of p-competition graphs, a related notion plays an analogous role. Suppose G is a graph and $F = \{S_1, \ldots, S_r\}$ is a family of subsets of the vertex set of G, repetitions allowed. We say that F is a *p-edge clique covering* or *p-ECC* if for every set $\{i_1, i_2, \ldots, i_p\}$ of p distinct subscripts, $T = S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_p}$ either is empty or induces a clique of G, and the collection of sets of the form T covers all edges of G. Note that a 1-ECC is an ordinary ECC. (An equivalent notion of "p-generator" is defined in Chung and West [6].) Let $\theta_e^p(G)$ be the smallest r for which there is a p-ECC. Our first result is analogous to the Dutton-Brigham theorem [9] for competition graphs of arbitrary digraphs.

THEOREM 1. *A graph G with n vertices is a p-competition graph of an arbitrary digraph if and only if $\theta_e^p(G) \le n$, which is true if and only if G has a p-*ECC *consisting of n sets.*

Proof. Suppose $G = C_p(D)$, where $D = (V, A)$, and let $V(G) = \{v_1, \ldots, v_n\}$. For each i, let $S_i = \{v_j : (v_j, v_i) \in A\}$, making the family of S_i a p-ECC. Conversely, suppose G and a p-ECC $F = \{S_1, \ldots, S_r\}$, with $r \le n$, are given. Define $D = (V, A)$ on $V = V(G)$ by letting $(v_i, v_j) \in A$ if and only if $v_i \in S_j$. It is easy to verify that $G = C_p(D)$. The last part of the theorem follows because repetitions are allowed in F. ∎

Note that the D constructed in the proof of Theorem 1 might indeed have loops. Note also that repetitions need to be allowed in F, since K_2 is a 2-competition graph but does not have a 2-ECC which consists of two distinct sets of vertices. Furthermore, repetitions cannot be allowed among the subscripts $i_1, \ldots, i_p$, since the 4-cycle C_4 is not a 2-competition graph (as shown in the following corollary), but the collection of two-element sets corresponding to its edges would form a 2-ECC if repetitions were allowed among the i's.

COROLLARY 1. *C_4 is not a 2-competition graph of an arbitrary digraph.*

Proof. Let the vertices around the 4-cycle be, in order, x_1, x_2, x_3, x_4. By Theorem 1, we can suppose F is a 2-ECC which consists of four sets. Suppose that some vertex, say x_1, belongs to all four sets. Since x_2, x_3 appear together in at least two sets of F, x_1 and x_3 must both appear together in at least two sets, which contradicts the nonadjacency of x_1 and x_3. Thus, each x_i is in at most three sets in F. But x_1 must be in at least two sets with x_2 and at least two sets with x_4, while x_2 and x_4 are in at most one set together. Thus, x_1 is in at least three sets, and so in exactly three sets of F. We conclude that each x_i is in exactly three sets of F. But then x_1 and x_3 are in two sets together, which is a contradiction. ■

Corollary 2 will be stated in terms of $\theta_e(G)$, the cardinality of a smallest ECC of G, and will give a condition for G to be a p-competition graph of an arbitrary digraph. It is a consequence of Theorem 1 and the following lemma.

LEMMA 2. *For any graph G, $\theta_e^p(G) \leq \theta_e(G) + p - 1$.*

Proof. Let $L_1, \ldots, L_r$ be an ECC of G. Define $S_i = L_i$ if $i \leq r$ and $S_i = V(G)$ if $i = r+1, \ldots, r+p-1$. To see that the S_i's form a p-ECC, note that the intersection of any p of them is either empty or the intersection of some of the L_i's, and hence is empty or a clique. Also, any edge of G is included in some L_i and hence in $S_i \cap S_{r+1} \cap \cdots \cap S_{r+p-1}$. ■

COROLLARY 2. *If G has n vertices and $\theta_e(G) \leq n - p + 1$, then G is a p-competition graph of an arbitrary digraph.*

Recall that a graph is a *chordal graph* whenever every cycle of length at least four has a chord (an edge joining two nonconsecutive vertices along the cycle).

COROLLARY 3. *Every chordal graph is a 2-competition graph of an* *arbitrary digraph.*

Proof. If G is chordal, then $\theta_e(G) \leq n-1$. The proof is by induction on n, noting that, by [8], every chordal graph G has a vertex x whose closed neighborhood is a clique, and so $\theta_e(G) \leq \theta_e(G-x)+1$. ∎

The bound in Corollary 2 is sharp, since C_4 is not a 2-competition graph of an arbitrary digraph and $\theta_e(C_4) = n-p+2$ if $p=2$. The 6-cycle C_6 shows that the converse is not true: $\theta_e(C_6) = 6 > n-p+1$ if $p=2$, but by the corollary to Theorem 3 below, C_6 is a 2-competition graph. We also show in Theorem 4 that the converse of Corollary 2 fails: $\theta_e(G)$ can be arbitrarily larger than $n-p+1$ for G a p-competition graph, indeed even when $p=2$. We next give a few results about 2-competition graphs in preparation for Theorem 4.

We first show that several simple constructions allow us to build up a new 2-competition graph from another 2-competition graph G. One construction is that of *adding a pendant edge* at a vertex x of G. This means adding to G a new vertex u and a new edge $\{x,u\}$. Another construction is a restricted case of *adding a path* P_k between vertices x and y of G. This means adding to G vertices $x_1, x_2, \ldots, x_{k-2}$ and edges $\{x, x_1\}$, $\{x_1, x_2\}, \ldots, \{x_{k-3}, x_{k-2}\}$, $\{x_{k-2}, y\}$. A third construction is a restricted case of *adding a path* P_k *with a pendant edge*, which means adding a path P_k as above and also a vertex v and edge $\{x_i, v\}$ for some i between 1 and $k-2$.

THEOREM 3. *Suppose G is a 2-competition graph of an arbitrary digraph. Then G' is a 2-competition graph of an arbitrary digraph if G' is obtained from G by*

(a) *adding a pendant edge at a vertex;*
(b) *adding a path P_k with $k=3$ between adjacent vertices x and y in G, or adding P_k with $k>4$ between arbitrary distinct vertices in G; or*
(c) *adding a path P_4 from x to y with a pendant edge, where x and y are distinct vertices in G.*

Proof. Let G have n vertices. By Theorem 1, there is a 2-ECC $S_1, \ldots, S_n$ for G.

(a): Let G' be obtained from G by adding a pendant edge $\{x,u\}$ at vertex x. We may assume that x is in S_1. For, if x is in any S_i, we may assume it is in S_1. If x is not in any S_i, then we can add x to S_1 without changing the fact that we have a 2-ECC. Form a 2-ECC for G' using $n+1$ sets by setting $S_1' = S_1 \cup \{u\}$, $S_i' = S_i$ for $i = 2, \ldots, n$, and $S_{n+1}' = \{x,u\}$.

(b): Let G' be obtained from G by adding a path P_k with $k = 3$ or $k > 4$ from x to y. Suppose $k > 4$. We shall show that we can assume without loss of generality that $x \in S_1$ and $y \in S_2$. If $\{x, y\}$ is an edge of G, then x, y are together in two sets S_i and S_j, and without loss of generality these are S_1 and S_2. Suppose $\{x, y\}$ is not an edge. As in the proof of part (a), we may assume without loss of generality that x is in S_1. If y is in any S_j, $j \neq 1$, we may assume y is in S_2. If y is not in any S_j, $j = 1, \ldots, n$, we may add y to S_2 without changing the fact that $S_1, \ldots, S_n$ is a 2-ECC. If y is only in S_1 and x is also in some S_j, $j \neq 1$, we may switch subscripts to get the desired conclusion. It remains to consider the case where x and y are in S_1, but neither x nor y is in any other S_j. Then we may remove y from S_1 and add it to S_2 without changing the fact that we have a 2-ECC.

Now, we build a 2-ECC for G' which uses $n + k - 2$ sets by taking $S_1' = S_1 \cup \{x_1\}$, $S_2' = S_2 \cup \{x_{k-2}\}$, $S_i' = S_i$ for $i = 3, \ldots, n$, $S_{n+1}' = \{x, x_1, x_2\}$, $S_{n+2}' = \{x_1, x_2, x_3\}$, $S_{n+3}' = \{x_2, x_3, x_4\}, \ldots, S_{n+k-2}' = \{x_{k-3}, x_{k-2}, y\}$. (Note that if this construction were used for $k = 4$, then $\{x, x_2\}$ could be in the two sets S_2' and S_{n+1}', which is not acceptable.)

Suppose $k = 3$, and that $\{x, y\}$ is an edge of G. In this case, there are two sets, S_1 and S_2 without loss of generality, which both contain x and y. Build a 2-ECC for G' by taking $S_1' = S_1 \cup \{x_1\}$, $S_i' = S_i$, $i = 2, \ldots, n$, and $S_{n+1}' = \{x, x_1, y\}$.

(c): Let G' be obtained from G by adding a path P_4 from x to y and a pendant edge, without loss of generality $\{x_2, v\}$. If x is not in any S_i, then we may place x in S_1 and not change the fact that we have a 2-ECC. Thus, by changing subscripts if necessary, we may assume that x is in S_1.

Assume first that y is not in S_1. Then, define a 2-ECC for G' by taking $S_1' = S_1 \cup \{x_1\}$, $S_i' = S_i$, $i = 2, \ldots, n$, $S_{n+1}' = \{x_1, x_2, y\}$, $S_{n+2}' = \{x, x_1, x_2, v\}$, $S_{n+3}' = \{y, x_2, v\}$. (Note that to prove that this is a 2-ECC, we need y not in S_1, for otherwise y, x_1 are in two sets together.)

Suppose next that x and y are both in S_1. Define a 2-ECC for G' by taking $S_1' = S_1 \cup \{x_1, x_2\}$, $S_i' = S_i$, $i = 2, \ldots, n$, $S_{n+1}' = \{x, x_1\}$, $S_{n+2}' = \{y, x_2, v\}$, $S_{n+3}' = \{x_1, x_2, v\}$. ∎

Note that part (b) of Theorem 3 would be false with $k = 4$, since K_2 is a 2-competition graph but C_4 is not. Part (b) would also be false with $k = 3$ and x and y nonadjacent, since P_3 is a 2-competition graph (by Corollary 3 to Theorem 1) but C_4 is not.

Recall that a *unicyclic graph* is a graph which is connected and has exactly one cycle.

COROLLARY 4. *All unicyclic graphs except C_4 are 2-competition graphs of arbitrary digraphs.*

Proof. Note that K_4 is a 2-competition graph. Each n-cycle C_n with $n \neq 4$ is a 2-competition graph by Theorem 3(b). Suppose that G is unicyclic and that the unique cycle in G has length n. If $n \neq 4$, then G can be built up from C_n by adding pendant edges, and so is a 2-competition graph by Theorem 3(a). If $n = 4$ and $G \neq C_4$, then G can be built up from C_4 by successively adding pendant edges, and so is a 2-competition graph by Theorem 3(a) [noting by Theorem 3(c) that C_4 with a pendant edge is a 2-competition graph, since it can be constructed from K_2 by adding a P_4 with a pendant edge]. ■

Isaak et al. [13] study the question of what complete bipartite graphs are 2-competition graphs. One of the results shown there is that $K_{2,x}$ is a 2-competition graph if and only if $x = 1$ or $x \geq 9$.

We can now show that the converse of Corollary 2 to Theorem 1 can be wrong in an arbitrarily large way.

THEOREM 4. *For every t, there is a graph G which is a p-competition graph of an arbitrary digraph and which has*

$$\theta_e(G) - (n - p + 1) > t.$$

Proof. Start with C_6, which is a 2-competition graph by Corollary 4. For C_6, $\lambda = \theta_e(G) - (n - p + 1) = 1$ if $p = 2$. Adding a path P_k with $k > 4$ between two distinct vertices of C_6 produces another 2-competition graph by Theorem 3(b). Moreover, θ_e has increased by $k - 1$, and the number of vertices n has increased by $k - 2$, which means that $\lambda = 2$. Repeat the process, each time adding a path P_k with $k > 4$ between two distinct vertices, and so each time increasing λ by 1. ■

3. p-COMPETITION GRAPHS OF ARBITRARY LOOPLESS DIGRAPHS AND OF ACYCLIC DIGRAPHS

In the case of ordinary competition graphs, Roberts and Steif [35] prove that K_2 is the only graph which is a competition graph of a digraph with loops but which is not one when loops are forbidden. The situation is more complicated for p-competition graphs. Since any graph of at most $p + 1$ vertices which has an edge is obviously not a p-competition graph of a loopless digraph, Theorem 1 or the corollary to Theorem 3 can be used to see that K_2, $K_2 \cup K_1$, P_3, and K_3 are 2-competition graphs but not if the digraph is required to be loopless. For every $p \geq 1$, K_{p+1} is an example of a p-competition graph which is not a p-competition graph of a loopless digraph.

The following theorem is analogous to the result of Dutton and Brigham [9] about ordinary competition graphs and has an analogous proof using Theorem 1.

THEOREM 5. *Suppose G is a graph of n vertices. Then G is a p-competition graph of a loopless digraph if and only if G has a p-ECC consisting of sets $S_1, \ldots, S_n$ and a labeling of vertices as $v_1, \ldots, v_n$ so that $v_i \in S_j$ implies $i \neq j$.*

The next result is analogous to one of Roberts and Steif [35] about ordinary competition graphs of loopless digraphs and has an analogous proof using Theorem 5.

LEMMA 6. *A graph G is the p-competition graph of a loopless digraph if and only if G has a p-ECC $S_1, \ldots, S_q$ such that if $D_i = V(G) - S_i$, then $\{D_1, D_2, \ldots, D_q\}$ has a system of distinct representatives.*

THEOREM 7. *Suppose G is a graph of n vertices. Then G is a p-competition graph of a loopless digraph if and only if G has a p-ECC $S_1, \ldots, S_q$ such that for all $k \leq q$, $|S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_k}| \leq n - k$.*

Proof. By Philip Hall's theorem, $\{D_1, \ldots, D_q\}$ has a system of distinct representatives if and only if the condition in the theorem holds. ∎

Note that Theorem 7 is not as complete a result as the result of [35] which lists all examples of graphs which are 1-competition graphs of arbitrary digraphs but not of loopless digraphs, K_2 being the only example. We have not been able to compile a complete list of such examples, even for the case $p = 2$.

In studying competition graphs of acyclic digraphs, Roberts [32] observed that adding sufficiently many isolated vertices to an arbitrary graph G makes it into the competition graph of some acyclic digraph. The smallest such number of isolated vertices was called the *competition number* of G and denoted $k(G)$. The characterization of competition graphs of acyclic digraphs reduced to the question of computing the competition number of an arbitrary graph. Analogously, Kim et al. [21] show that every graph G can be made into the p-competition graph of some acyclic digraph by adding sufficiently many isolated vertices, and they call the smallest such number of isolated vertices the *p-competition number* of G, $k_p(G)$. In particular, [21] shows that, for all graphs G, $k_p(G) \leq k(G) + p - 1$, and that in many cases this bound is tight; nevertheless, there are G for which $k_p(G) < k(G)$, and even when $p = 2$ the difference can be made arbitrarily large.

The next theorem is a graph-theoretic characterization which is analogous to those for competition graphs by Dutton and Brigham [9] and Lundgren and Maybee [24]. The proof is also analogous, using Theorem 5.

THEOREM 8. *Suppose G is a graph of n vertices. Then G is a p-competition graph of an acyclic digraph if and only if G has a p-ECC consisting of sets $S_1, \ldots, S_n$ and a labeling of vertices as $v_1, \ldots, v_n$ so that $v_i \in S_j$ implies $i < j$.*

4. CLOSING REMARKS

The results in this paper leave some natural questions unresolved. For example, we have not been able to find a complete list of graphs which are p-competition graphs for arbitrary digraphs with loops but not for arbitrary digraphs without loops. Also, are there interesting families of graphs which are 3-competition graphs of arbitrary digraphs?

We have also not touched upon the natural generalization of the major open problem for ordinary competition graphs: Which acyclic digraphs have p-competition graphs which are interval graphs? A related problem is to characterize those p-competition graphs which are interval graphs under different assumptions about the underlying acyclic digraphs. (See [12] for recent work on these latter two problems for the case $p = 1$.)

Another direction of research would be to apply the theory of p-ECCs which we have developed to study p-generalizations of various classes of graphs which have characterizations by edge clique coverings. A variety of such classes of graphs are described in the survey paper [34]. To give an example, if F is a family of sets, its *p-intersection graph* can be defined to be the graph whose vertices are the sets in F and whose edges correspond to pairs of sets which have at least p elements in common. Thus, the 1-intersection graph of F is the same as the ordinary intersection graph. The analogy between intersection graphs and p-intersection graphs is studied in [28].

It can easily be shown that every graph arises as a p-intersection graph for every p (see [16]). Moreover, we can define the *p-intersection number* of a graph G to be the cardinality of the smallest set S such that G is a p-intersection graph of a family of subsets of S. Then one can show that this p-intersection number is the same as $\theta_e^p(G)$, which is a generalization of the well-known result that the ordinary intersection number of G is the same as $\theta_e(G)$; see [34]. From Theorem 1 we see that determining p-intersection numbers is important in the study of p-competition graphs. This parameter has been recently investigated in [4, 6, 14, 17, 20, 21].

To give a second example, a topic of interest related to competition graphs is the study of upper bound graphs of posets; see [25, 29, 30]. If $\langle X, < \rangle$ is a poset, its *upper bound graph* G is defined as follows: $V(G) = X$, and $\{u, v\} \in E(G)$ if any only if, for some $w \in X$, either $u < w$ or $u = w$, and also either $v < w$ or $v = w$. Analogously, we might say that G is the *p-upper bound graph* of $\langle X, < \rangle$ if $V(G) = X$ and there are distinct $w_1, \ldots, w_p \in X$ such that, for $i = 1, \ldots, p$, either $u < w_i$ or $u = w_i$ and also either $v < w_i$ or $v = w_i$. The following characterization of p-upper bound graphs is analogous to the characterization of upper bound graphs using ordinary ECC's in [25] and has an analogous proof.

THEOREM 9. *Suppose G is a graph of n vertices. Then G is a p-upper bound graph of a poset if and only if G has a p-ECC consisting of sets $S_1, \ldots, S_n$ and a labeling of vertices as $v_1, \ldots, v_n$ such that $v_i \in S_i$ for all i and also such that $(v_i \in S_j)$ implies both $i \leq j$ and $S_i \subseteqq S_j$ for all i, j.*

Other classes of graphs, such as neighborhood graphs and 2-step graphs, defined in [34], are studied in [27].

Suh-ryung Kim and Fred S. Roberts thank the Air Force Office of Scientific Research for its support under grants AFOSR-85-0271 and AFOSR-89-0066 to Rutgers University. Terry A. McKee thanks the Office of Naval Research for its support under grant N00014-88-K-0163 to Wright State University. F. R. McMorris thanks the Office of Naval Research for its support under grant N00014-89-J-1643 to the University of Louisville. The authors thank Garth Isaak, Denise Sakai and Chi Wang for suggesting improvements and other useful comments.

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Received 20 July 1993; final manuscript accepted 27 February 1994